Chaotic Modeling and Simulation (CMSIM) 4: 633–642, 2012

Delay Factors and Chaotization of Non-ideal Pendulum Systems

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Abstract. The oscillations of a plane pendulum, the suspension point of which excites by electric motor of the limited power with taking into account various factors delay are considered. Regular and chaotic attractors, phase-parametric characteristics and maps of dynamic regimes of this system are built and explored. Influence of delay factors on the appearance of deterministic chaos is analysed in detail.
Keywords: delay, chaotic attractor, map of dynamic regimes.

1 Introduction

Pendulum mathematical models are widely used to describe the dynamics of various oscillatory systems. Such models are used to study the oscillation of free liquid surface, membranes, various technical constructions, in the study of cardiovascular system of live organisms, financial markets, etc.

The problems of global energy saving require the highest minimization of excitation source power of oscillatory systems. This leads to the fact that the energy of excitation source is comparable to the energy consumed by the oscillating system. Such systems as "source of excitation - oscillating subsystem" are called nonideal by Zommerfeld-Kononenko [1]. In mathematical modeling of such systems, the limitation of excitation source power must be always taken into account.

Another important factor, that significantly have an influence on steady-state regimes of dynamical systems are different, by their physical nature, factors of delay. The delay factors are always present in rather extended systems due to the limitations of signal transmission speed: waves of compression, stretching, bending, current strength, etc. The study of the influence of delay factors on the dynamical stability of equilibrium positions of pendulum systems was initiated by Yu. A. Mitropolsky and his scientific school in the 80s of the last century [2], [3]. But only ideal pendulum models were initially considered. Mathematical models of pendulum system with limited excitation, taking into account the influence of different factors of delay, were first obtained in [4], [5]. The influence of delay factors on existence and dynamic stabilization of

Received: 24 August 2012 / Accepted: 23 October 2012

ISSN 2241-0503

pendulum equilibrium positions at limited excitation was studied. Later it was discovered the existence of chaotic attractors in nonideal systems "pendulum - electric motor" and proved that the main cause of chaos is limited excitation [4], [5], [7].

In this paper the oscillations of a flat pendulum, the suspension point of which excites by electric motor of the limited power with taking into account various factors delay are considered.

2 Mathematical model of the system

In [6] the equations of motion of the system "pendulum - electric motor" in the absence of any delay factors were obtained. They are

$$\begin{cases} \dfrac{dy_1}{d\tau} = Cy_1 - y_2y_3 - \dfrac{1}{8}(y_1^2y_2 + y_2^3); \\ \dfrac{dy_2}{d\tau} = Cy_2 + y_1y_3 + \dfrac{1}{8}(y_1^3 + y_1y_2^2) + 1; \\ \dfrac{dy_3}{d\tau} = Dy_2 + Ey_3 + F; \end{cases} \tag{1}$$

where phase variables y_1y_2 describe the pendulum deviation from the vertical and phase variable y_3 is proportional to the rotation speed of the motor shaft. The system parameters are defined by

$$C = -\delta\varepsilon^{-2/3}\omega_0^{-1}, D = -\frac{2ml^2}{I}, F = 2\varepsilon^{-2/3}(\frac{N_0}{\omega_0} + E) \tag{2}$$

where m - the pendulum mass, l - the reduced pendulum length, ω_0 - natural frequency of the pendulum, a - the length of the electric motor crank, $\varepsilon = \frac{a}{l}$, δ - damping coefficient of the medium resistance force, I - the electric motor moment of inertia, E, N_0 - constants of the electric motor static characteristics.

Let us consider the following system of equations:

$$\begin{cases} \dfrac{dy_1(\tau)}{d\tau} = Cy_1(\tau-\delta) - y_2(\tau)y_3(\tau-\gamma) - \dfrac{1}{8}(y_1^2(\tau)y_2(\tau) + y_2^3(\tau)); \\ \dfrac{dy_2(\tau)}{d\tau} = Cy_2(\tau-\delta) + y_1(\tau)y_3(\tau-\gamma) + \dfrac{1}{8}(y_1^3(\tau) + y_1(\tau)y_2^2(\tau)) + 1; \\ \dfrac{dy_3(\tau)}{d\tau} = Dy_2(\tau-\gamma) + Ey_3(\tau) + F; \end{cases} \tag{3}$$

This system is a system of equations with constant delay. Positive constant parameter γ was introduced to account the delay effects of electric motor impulse on the pendulum. We assume that the delay of the electric motor response to the impact of the pendulum inertia force is also equal to γ. Taking into account the delay γ conditioned by the fact that the wave velocity perturbations on the elements of the construction has a finite value that depends on the properties of external fields, for instance, the temperature field. In turn, the constant positive parameter δ characterizes the delay of the medium reaction on the dynamical state of the pendulum. This delay is due to the limited sound velocity in that medium.

Assuming a small delay, we can write

$$y_1(\tau-\delta)=y_1(\tau)-\frac{y_1(\tau)}{d\tau}\delta+...,\quad y_2(\tau-\delta)=y_2(\tau)-\frac{y_2(\tau)}{d\tau}\delta+...$$

$$y_2(\tau-\gamma)=y_2(\tau)-\frac{y_2(\tau)}{d\tau}\gamma+...,\quad y_3(\tau-\gamma)=y_3(\tau)-\frac{y_3(\tau)}{d\tau}\gamma+...$$

Then, if $C\delta\neq -1$, we get the following system of equations:

$$\begin{cases} \dot{y_1}=\dfrac{1}{1+C\delta}\left(Cy_1-y_2\left[y_3-\gamma\left(Dy_2+Ey_3+F\right)\right]-\dfrac{1}{8}(y_1^2y_2+y_2^3)\right); \\ \dot{y_2}=\dfrac{1}{1+C\delta}\left(Cy_2+y_1y_3-y_1\gamma(Dy_2+Ey_3+F)+\dfrac{1}{8}(y_1^3+y_1y_2^2)+1\right); \\ \dot{y_3}=(1-C\gamma)Dy_2-\dfrac{D\gamma}{8}(y_1^3+y_1y_2^2+8y_1y_3+8)+Ey_3+F. \end{cases} \tag{4}$$

The obtained system of equations is already a system of ordinary differential equations. Delays are included in this system as additional parameters.

The study of the influence of delay factors on existence and stability of equilibrium positions of the system (4) was carried out. Also the effect of delay on origin the deterministic chaos was studied. Since the system of equations (4) is strongly nonlinear, the study of steady-state regimes has been carrying out using numerical methods. A large set of computer experiments have been held to determine the possibility of chaotic attractors' occurrence in the system (4). The methodology of these computer experiments is described in detail in [6].

3 Dynamic regimes maps

A very clear picture of the dynamical system behavior can give us a map of dynamic regimes. It is a diagram on the plane, where two parameters are plotted on axes and the boundaries of different dynamical regimes areas are shown. The construction of dynamical regimes maps is based on the analysis and processing of spectrum of Lyapunov characteristic exponents, phase portraits, Poincare sections and maps, Fourier spectrums and distributions of the invariant measure of attractors of the system. The atlas of maps of dynamic regimes of nonideal system "pendulum–electric" motor in the absence of delay factors were obtained in [8].

Fig. 1a shows a map of dynamical regimes, when the parameters E and D are changing, and the parameters C, D are equal to $C=-0.1, F=0.19$ in the case of delay absence in the system. The dark-grey areas of the maps correspond to equilibrium positions of the system. The light-grey areas of the maps correspond to limit cycles of the system. And finally, the black areas of the maps correspond to chaotic attractors.

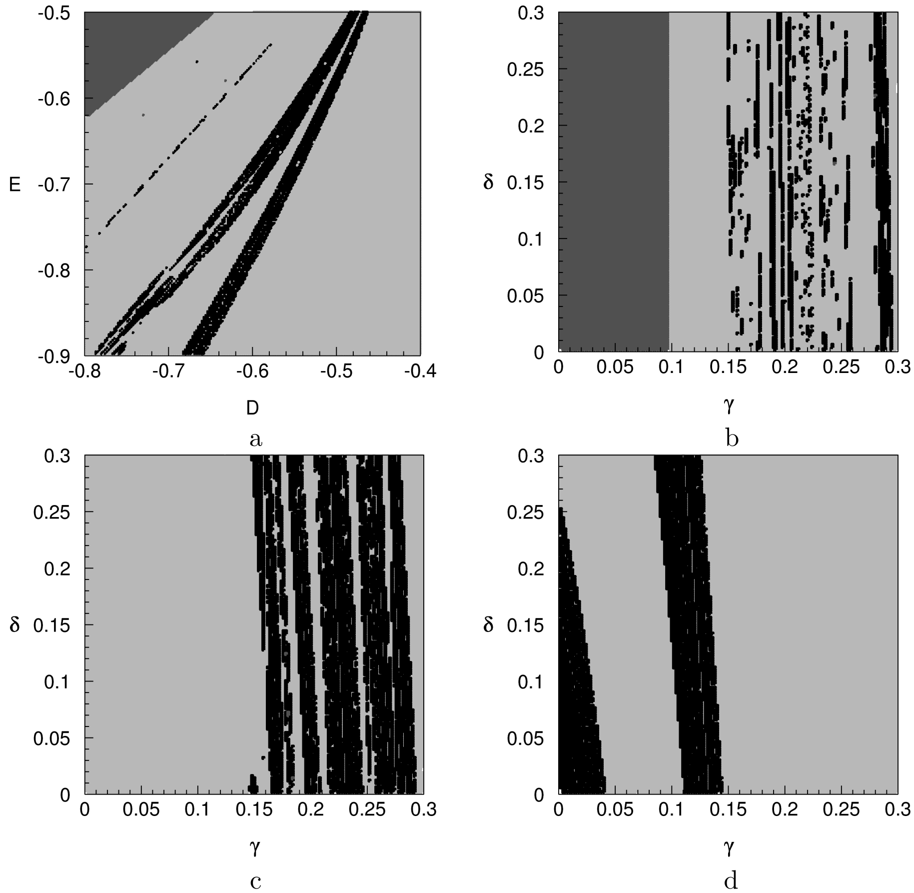

Fig. 1. Maps of dynamic regimes

Fig. 1b–d illustrate the influence of delays γ and δ on changing the type of steady-state regime of the system. Thus Fig.1b was built at the values of the parameters $D = -0.8, E = -0.6$ that correspond to the dark-grey area of the map 1a. As can be seen from Fig.1b the type of steady-state regime does not change at very small values of the delay. It is still an equilibrium position. However, with an increase of delay values γ and δ the limit cycles appear in the system (light-grey areas of the map) and then the chaotic attractors (black areas of the map). And there is a quite complex and fanciful structure of alternating regions of existence of periodic and chaotic regimes.

Fig.1c was built at the values of the parameters $D = -0.58, E = -0.6$ that correspond to the light-grey area of the map 1a (periodic regimes). As can be seen from this figure the attractor of the system is limit cycle at very small values of the delays. With an increase of the delay values the chaos arises in the system (black areas in the figure). Finally, fig.1d was built at the values of the parameters $D = -0.53, E = -0.6$ that correspond to the black area of the map 1a (chaotic regimes). In this case, with an increase of the delay values the region of chaos is replaced by the region of periodic regimes. Then again chaos arises in the system.

Apparently from figures 1b–d when the delay of interaction between a pendulum and the electromotor γ greater than certain value, change of types of steady-state regimes can be observed at very small value of delay of a medium δ.

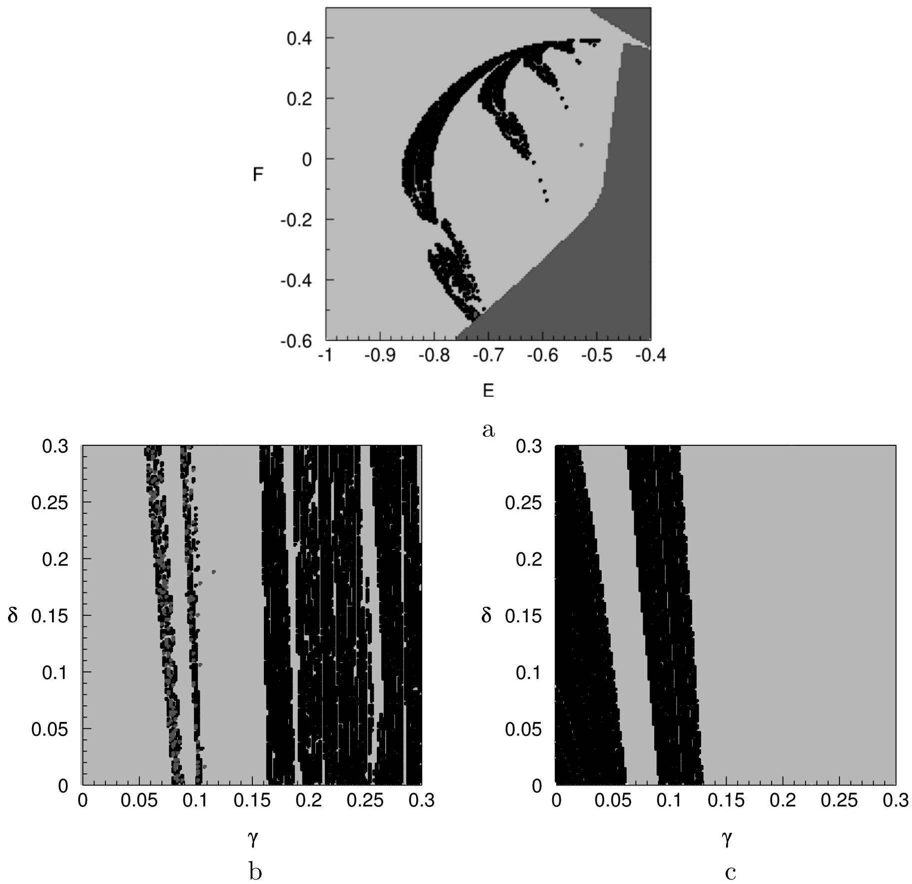

Fig. 2. Maps of dynamic regimes

In fig.2a the map of dynamical regimes at $C = -0.1, D = -0.6$ in the case of delay absence is constructed. In fig.2b the change of steady-state regimes types, which takes place in the system with an increase of the delay values is shown. Thus the point (0, 0) of the map corresponds to the values of the parameters $E = -0.7, F = -0.4$ at which the steady-state regime is a limit cycle. With an increase of the delay values in the map there are narrow areas in which the limit cycle is replaced by an equilibrium position, as well as by a chaotic attractor. Further there is a rather wide area of periodic regimes, which with further increase of the delay is replaced by chaos area. But, in this rather wide area of chaos narrow strips of periodic regimes "are built in". A Fig.2c was built for parameters $E = -0.67, F = 0.3$ that corresponds to the black area of the map 2a (chaotic regimes). Here, with an increase of values of delay the changes of the steady-state regime such as "chaos–cycle–chaos–cycle" are observed. And, apparently from this figure, we see that change of type of a steady-state regime can be observed and at very small values of the delays. increase of the delay happens under the scenario of Pomeau-Manneville, in a single bifurcation, through intermittency.

Similar changes of the dynamical regimes types of the system (4), which are observed with the delay change, are shown in Fig.3. The initial map of dynamical regimes was built at $C = -0.1$, $E = -0.59$ (fig.3a). The maps in fig. 3b, c are constructed at $D = -0.53$, $F = -0.4$ and $D = -0.5$, $F = -0.31$ respectively. As can be seen from fig. 3b, c with an increase of the delay values the change of the steady-state regime may take place such as "equilibrium position–cycle–chaos" or "cycle–chaos" in different variations of the order of these changes.

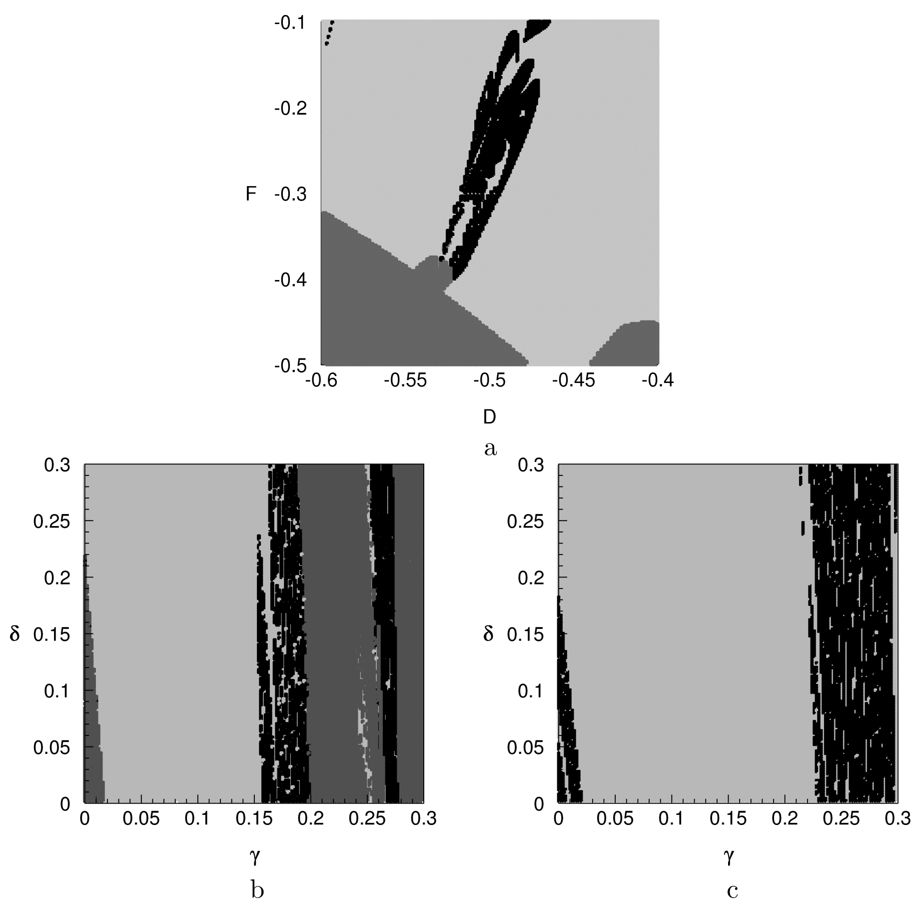

Fig. 3. Maps of dynamic regimes

4 The study of steady-state regimes of interaction

Let us consider in detail the types of regular and chaotic attractors that exist in the system (4). Let us consider the behavior of the system (4) when parameters are $C = -0.1$, $D = -0.8$, $E = -0.6$, $F = 0.19$ and the delays $\gamma = 0.29$ and $0 \leq \delta \leq 0.29$. In fig. 4a,b the dependence of maximum non-zero Lyapunov's characteristic exponent and phase-parametric characteristic of the system are shown respectively. These figures illustrate the influence of the delay of the medium δ, in which the oscillations of the pendulum are, on chaotization of the system (4).

So in Fig.4a we can clearly see the presence of intervals δ in which maximum Lyapunov exponent of the system is positive. In these intervals in the system there are chaotic attractors. The area of existence of chaos is clearly seen in phase-parametric characteristic of the system. The areas of chaos in the bifurcation tree are densely filled with points. A careful examination of the obtained images allows not only to identify the origin of chaos in the system,

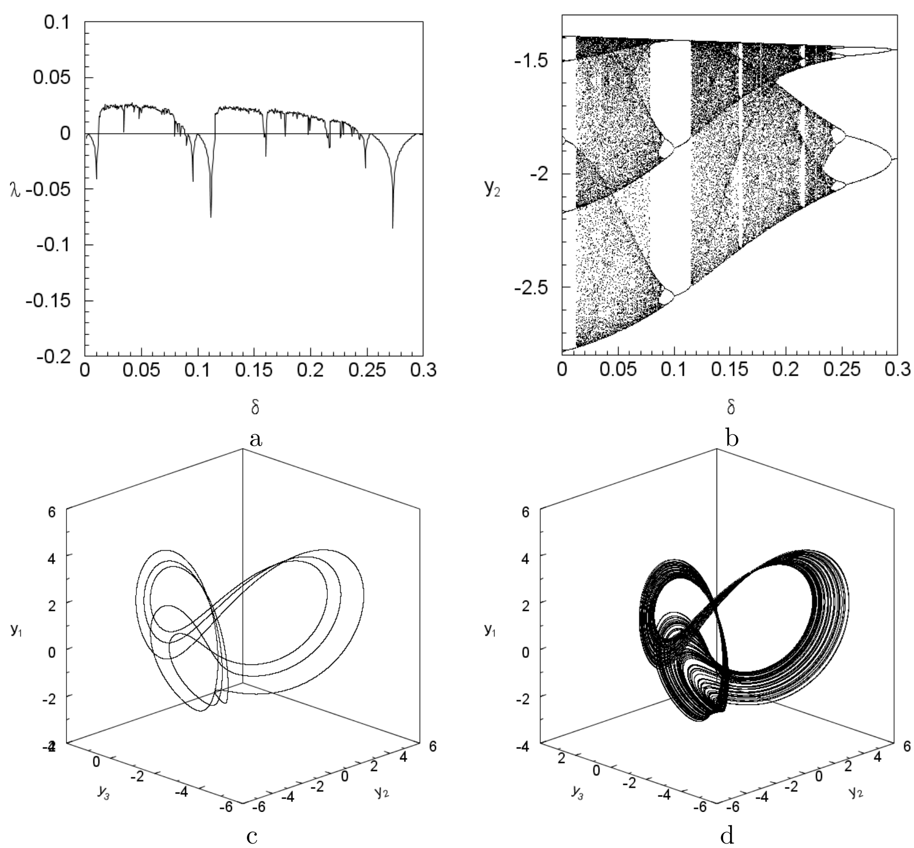

Fig. 4. The dependence of maximal non-zero Lyapunov's characteristic exponent (a), phase-parametric characteristic of the system (b), phase portrait of the limit cycle (c) and the chaotic attractor (d)

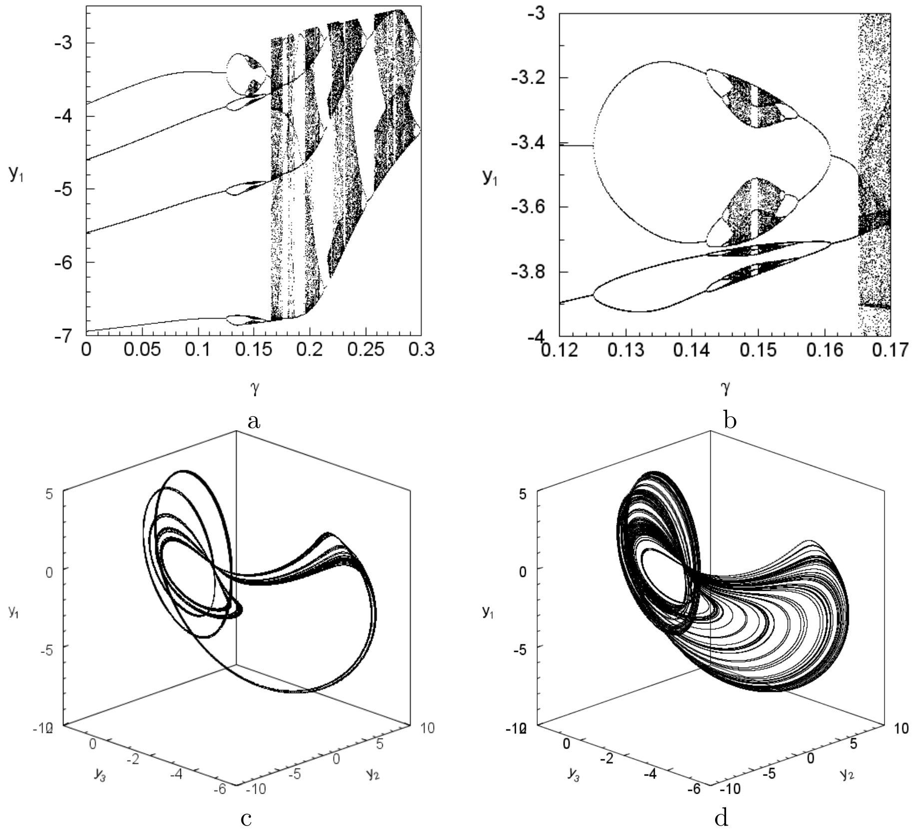

Fig. 5. Phase-parametric characteristics of the system (a)–(b), phase portraits of the chaotic attractors (c)–(d)

but also to describe the scenario of transition to chaos. So with a decrease of δ there are the transitions to chaos by Feigenbaum scenario (infinite cascade of period-doubling bifurcations of a limit cycle). Bifurcation points for delay δ are clearly visible in Fig.4a as well as in Fig.4b. These points are the points of approaches of the Lyapunov's exponent graph to the zero line (Fig.4a) and the points of splitting the branches of the bifurcation tree (Fig.4b). In turn, the transition to chaos with an increase of the delay happens under the scenario of Pomeau-Manneville, in a single bifurcation, through intermittency.

In Fig.4c,d phase portrait of one of the limit cycles, at $\delta = 0.11$, of the cascade of period-doubling bifurcations and phase portrait of the chaotic attractor at $\delta = 0.07$ that arises at the end of this stage are shown respectively.

Let us study the influence of the delay γ when parameters are $C = -0.1$, $D = -0.58$, $E = -0.6$, $F = 0.19$ and in the absence of the delay of the medium δ. In Fig.5a the phase-parametric characteristic of the system and in Fig.5b an enlarged fragment of the central part of this characteristic are constructed. Let us consider in more detail Fig.5b. Here there is situation, atypical for dynamical systems, of transition to chaos. As it is known, the most typical

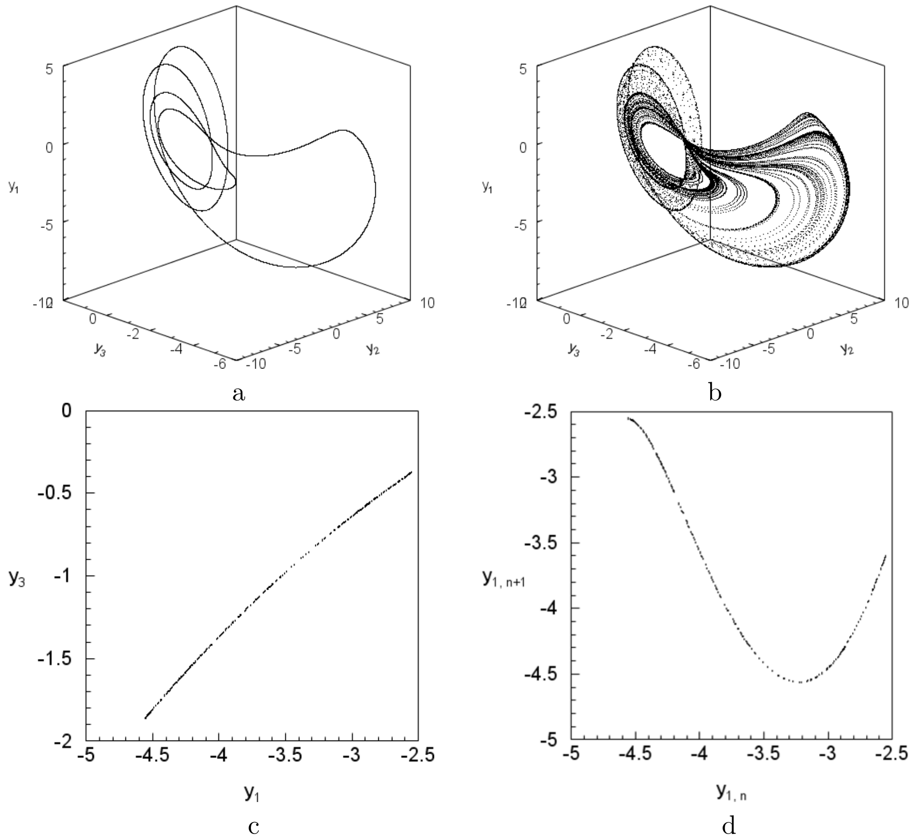

Fig. 6. The distribution of the invariant measure of the limit cycle (a) and the chaotic attractor (b), the Poincare section (c) and map (d) of the chaotic attractor

situation is when the transition to chaos happens with a decrease (increase) the bifurcation parameter through a cascade of period-doubling bifurcations and with an increase (decrease) the bifurcation parameter - through intermittency. Here there is some symmetry of scenarios of transition to chaos. As can be seen from Fig.5b, there is an interval of change the values γ in which the transition to chaos under the Feigenbaum scenario can be observed both with a decrease and with an increase of values γ. An analogous situation occurs in this interval of changes γ for intermittency. In other words the transition to chaos through intermittency can be observed both with a decrease and with an increase the values γ. On the right part of the bifurcation tree the typical situation for non-linear dynamics is observed.

In Fig.5c,d the typical in this case phase portraits of chaotic attractors of the system are shown. The chaos shown in Fig.5c is characterized by a relatively small volume of localization in phase space of the system (4). Conversely, in Fig.5d the "developed" chaos with a much larger volume of localization in phase space is shown.

Let the system (4) parameters be $C = -0.1$, $D = -0.58$, $E = -0.6$, $F = 0.19$ and the delay $\delta = 0$. The distributions of the invariant measure of the limit cycle at $\gamma = 0.165$ (fig.6a) and the chaotic attractor at $\gamma = 0.1652$ (fig.6b) illustrate the transition to chaos through intermittency when changing the delay γ. These distributions of the invariant measure on the phase portrait of attractors allow us to identify the laminar and turbulent phase of chaotic attractor that arises under the scenario of Pomeau-Manneville.

In fig. 6c,d, rather typical for this system, the Poincare section and the Poincare map of the chaotic attractor at $\gamma = 0.28$ are constructed. Both of them have "quasiribbon" structure. This allows building an analytical approximation of Poincare map that can be used for an approximate study the dynamics of the three-dimensional system using one-dimensional discrete map [9].

5 Conclusions

Thus, various factors of delay make the considerable influence on dynamics of system "pendulum - electromotor". Delay presence in such systems can cause both origin, and vanishing of chaotic attractors. Besides, delay leads to occurrence of atypical situations at transitions from the regular regimes to deterministic chaos.

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