Chaotic Modeling and Simulation (CMSIM) 1: 49–56, 2017

The influence of delay factors on the genesis of deterministic chaos in non-ideal pendulum systems

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Abstract. The dynamics of non-ideal dynamical system "pendulum electric motor" with taking into account various factors of delay is considered. Mathematical model of the system is a system of ordinary differential equations with delay. The approaches that reduce the mathematical model of this system to the system of three, nine or fifteen differential equations without delay are suggested. The influence of delay factors on steady-state regimes of the pendulum electric motor system is studied. Maps of dynamical regimes, the dependencies of maximal non-zero Lyapunov's characteristic exponent from the delay, phase-parametric characteristics are built and analyzed. The scenarios of transition from steady-state regular regimes to chaotic regimes are identified.

It is shown that neglecting the presence of delay in such systems may affect the qualitative change in their dynamics. In some cases the delay is the main reason of origination as well as vanishing of chaotic attractors. The use of three-dimensional mathematical model to study the dynamics of "pendulum electric motor" systems is sufficient only at small values of the delay. For relatively high values of the delay, multi-dimensional system of nine or fifteen equations should be used.

Keywords: pendulum system, regular and chaotic attractors, maps of dynamical regimes, scenarios of transition to chaos.

1 Introduction

In the majority of studies the dynamics of pendulum systems are being conducted without taking into account the limitations of excitation source power, so it is assumed that the power of excitation source considerably exceeds the power that consumes the vibrating system. Such systems are called ideal in sense of Sommerfeld-Kononenko [1]. In many cases such idealization leads to qualitative and quantitative errors in describing dynamical regimes of pendulum systems.

Received: 28 March 2016 / Accepted: 12 December 2016

ISSN 2241-0503

Modern development of energy efficient and energy-preserving technologies requires the highest minimization of excitation source power of oscillatory systems. This leads to the fact that the energy of excitation source is comparable to the energy consumed by the oscillating system. Such systems as “source of excitation-oscillating subsystem” are called non-ideal by Sommerfeld-Kononenko [1]. In mathematical modeling of such systems, the limitation of excitation source power must be always taken into account.

Another important factor that significantly affects the change of steady-state regimes of dynamical systems is the presence of different in their physical substance, factors of delay. The delay factors are always present in rather extended systems due to the limitations of signal transmission speed: stretching, waves of compression, bending, current strength, etc. In some cases, taking into account factors of delay leads only to minor quantitative changes in dynamic characteristics of pendulum systems. In other cases, taking into account these factors allow to identify qualitative changes in dynamic characteristics.

The study of the influence of delay factors on the dynamical stability of equilibrium positions of pendulum systems was initiated by Yu. A. Mitropolsky [2], [3]. Initially only ideal pendulum models were considered.

In this paper the oscillations of non-ideal pendulum systems of the type “pendulum-electric motor” with taking into account various factors of delay are considered. Mathematical models of pendulum system with limited excitation, taking into account the influence of different factors of delay, were first obtained in [4, 5]. The influence of delay factors on existence and dynamic stabilization of pendulum equilibrium positions at limited excitation was studied. Later the existence of chaotic attractors in nonideal systems ”pendulum - electric motor” was discovered and proved that the main cause of chaos is limited excitation [6], [7].

The aim of this paper is to study the influence of the delay of interaction between pendulum and electric motor and the delay of the medium in non-ideal pendulum systems of the type “pendulum-electric motor” on steady-state regular and chaotic regimes of oscillations.

2 Mathematical model of the system

Mathematical model of the dynamical system “pendulum-electric motor” in the absence of any delay factors can be written as the following system of equations [6]:

$$\begin{cases} \dfrac{dy_1}{d\tau} = Cy_1 - y_2 y_3 - \dfrac{1}{8}(y_1^2 y_2 + y_2^3); \\ \dfrac{dy_2}{d\tau} = Cy_2 + y_1 y_3 + \dfrac{1}{8}(y_1^3 + y_1 y_2^2) + 1; \\ \dfrac{dy_3}{d\tau} = Dy_2 + Ey_3 + F; \end{cases} \tag{1}$$

where phase variables $y_1 y_2$ describe the pendulum deviation from the vertical and phase variable y_3 is proportional to the rotation speed of the motor shaft.

The system parameters are defined by

$$C = -\delta_1 \varepsilon^{-2/3} \omega_0^{-1}, D = -\frac{2ml^2}{I}, F = 2\varepsilon^{-2/3}(\frac{N_0}{\omega_0} + E) \tag{2}$$

where m - the pendulum mass, l - the reduced pendulum length, ω_0 - natural frequency of the pendulum, a - the length of the electric motor crank, $\varepsilon = \frac{a}{l}$, δ_1 - damping coefficient of the medium resistance force, I - the electric motor moment of inertia, E, N_0 - constants of the electric motor static characteristics.

Let us consider the following system of equations [8]:

$$\begin{cases} \dfrac{dy_1(\tau)}{d\tau} = Cy_1(\tau - \delta) - y_2(\tau)y_3(\tau - \gamma) - \dfrac{1}{8}(y_1^2(\tau)y_2(\tau) + y_2^3(\tau)); \\ \dfrac{dy_2(\tau)}{d\tau} = Cy_2(\tau - \delta) + y_1(\tau)y_3(\tau - \gamma) + \dfrac{1}{8}(y_1^3(\tau) + y_1(\tau)y_2^2(\tau)) + 1; \\ \dfrac{dy_3(\tau)}{d\tau} = Dy_2(\tau - \gamma) + Ey_3(\tau) + F. \end{cases} \tag{3}$$

This system is a system of equations with constant delay. Positive constant parameter γ was introduced to account the delay effects of electric motor impulse on the pendulum. We assume that the delay of the electric motor response to the impact of the pendulum inertia force is also equal to γ. Taking into account the delay γ conditioned by the fact that the wave velocity perturbations on the elements of the construction has a finite value that depends on the properties of external fields, for instance, the temperature field. In turn, the constant positive parameter δ characterizes the delay of the medium reaction on the dynamical state of the pendulum. This delay is due to the limited sound velocity in that medium.

Let us consider two approaches that allow reducing the time-delay system (3) to the system of equations without delay. The first approach is as follows. Assuming a small delay, we can write

$$y_i(\tau - \gamma) = y_i(\tau) - \frac{y_1(\tau)}{d\tau}\gamma + ..., \quad i = 2, 3$$

$$y_i(\tau - \delta) = y_i(\tau) - \frac{y_1(\tau)}{d\tau}\delta + ..., \quad i = 1, 2$$

Then, if $C\delta \neq -1$, we get the following system of equations [8]:

$$\begin{cases} \dot{y_1} = \dfrac{1}{1 + C\delta}\left(Cy_1 - y_2\left[y_3 - \gamma\left(Dy_2 + Ey_3 + F\right)\right] - \dfrac{1}{8}(y_1^2 y_2 + y_2^3)\right); \\ \dot{y_2} = \dfrac{1}{1 + C\delta}\left(Cy_2 + y_1 y_3 - y_1\gamma(Dy_2 + Ey_3 + F) + \dfrac{1}{8}(y_1^3 + y_1 y_2^2) + 1\right); \\ \dot{y_3} = (1 - C\gamma)Dy_2 - \dfrac{D\gamma}{8}(y_1^3 + y_1 y_2^2 + 8y_1 y_3 + 8) + Ey_3 + F. \end{cases} \tag{4}$$

The obtained system of equations is already a system of ordinary differential equations. Delays are included in this system as additional parameters.

In order to approximate the system (3) another, more precise, method can be used [10], [11]. Let us divide each of the segments $[-\gamma;\ 0]$ and $[-\delta;\ 0]$ into m equal parts. We introduce the following notation

$$y_1(\tau - \tfrac{i\delta}{m}) = y_{1i}(\tau),\ y_2(\tau - \tfrac{i\gamma}{m}) = y_{2i}(\tau),\ y_2(\tau - \tfrac{i\delta}{m}) = \tilde{y}_{2i}(\tau),$$

$$y_3(\tau - \tfrac{i\gamma}{m}) = y_{3i}(\tau), i = \overline{0, m}.$$

Then, using difference approximation of derivative [10], [11] the system of equations with delay (3) can be reduced to the following system of equations without delay:

$$\begin{cases} \frac{dy_{10}(\tau)}{d\tau} = Cy_{1m}(\tau) - y_{20}(\tau)y_{3m}(\tau) - \frac{1}{8}(y_{10}^2(\tau)y_{20}(\tau) + y_{20}^3(\tau)); \\ \frac{dy_{20}(\tau)}{d\tau} = C\tilde{y}_{2m}(\tau) + y_{10}(\tau)y_{3m}(\tau) + \frac{1}{8}(y_{10}^3(\tau) + y_{10}(\tau)y_{20}^2(\tau)) + 1; \\ \frac{dy_{30}(\tau)}{d\tau} = Dy_{2m}(\tau) + Ey_{30}(\tau) + F; \\ \frac{dy_{1i}(\tau)}{d\tau} = \frac{m}{\delta}(y_{1\ i-1}(\tau) - y_{1i}(\tau)), \quad i = \overline{1, m}; \\ \frac{dy_{2i}(\tau)}{d\tau} = \frac{m}{\gamma}(y_{2\ i-1}(\tau) - y_{2i}(\tau)), \quad i = \overline{1, m}; \\ \frac{d\tilde{y}_{2i}(\tau)}{d\tau} = \frac{m}{\delta}(\tilde{y}_{2\ i-1}(\tau) - \tilde{y}_{2i}(\tau)), \quad i = \overline{1, m}; \\ \frac{dy_{3i}(\tau)}{d\tau} = \frac{m}{\gamma}(y_{3\ i-1}(\tau) - y_{3i}(\tau)), \quad i = \overline{1, m}. \end{cases} \tag{5}$$

Should be noted that the main variables in this system are only y_{10}, y_{20}, y_{30}. In other words the solutions y_1, y_2, y_3 of the system (3) are described by the functions y_{10}, y_{20}, y_{30} of the system (5).

The system (5) is a system of ordinary differential equations of $(4m+3)$-th order. Choosing a sufficiently large m in the system (5), the system (3) will be very well approximated by the system (5) [10]. In this paper the system of equation (5) was considered at $m = 3$. In this case, the system (5) has 15 equations. The calculations of cases $m > 3$, with a significant increase the number of equations, were also carried out. It was established, that increasing the number of equations has practically no effect on identification and description of steady-state regimes of "pendulum–electric motor" system. But it significantly increases the complexity of constructing characteristics, which are necessary for study the steady-state regimes of oscillations. Therefore, the use of mathematical model (5) at $m = 3$ is optimal for studying the influence of delay on regular and chaotic dynamics of "pendulum–electric motor" system.

Therefore, we obtained three-dimensional (4) and fifteen-dimensional (5) mathematical models each describing the system of equations with delay (3). These models are the systems of non-linear differential equations, so in general the study of steady-state regimes can be carried out only by using numerical methods and algorithms. The methodology of such studies is described in detail in [6].

3 Maps of dynamical regimes

For general analysis of nonlinear dynamics the maps of dynamical regimes are constructed. These maps provide a crucial information about the type of steady-state regime of the the system depending on its parameters. The construction of dynamical regimes maps is based on analysis and processing of spectrum of Lyapunov characteristic exponents [6]. Where necessary, for more accurate determination of steady-state regime of the system, we study other characteristics of attractors: Poincare sections and maps, Fourier spectrums, phase portraits and distributions of the invariant measure.

Let us consider the behavior of the systems (4) and (5) when the parameters are $C = -0.1$, $D = -0.53$, $E = -0.59$, $F = -0.4$. The map of dynamical regimes in fig. 1, a was built for three-dimensional model (4) and the map in fig. 1, b was built for fifteen-dimensional model (5). These figures illustrate the effect of the delay of interaction between pendulum and electric motor γ and the delay of the medium δ on changing the type of steady-state regime of the systems. The dark-grey areas of the maps correspond to equilibrium positions of the system. The light-grey areas of the maps correspond to limit

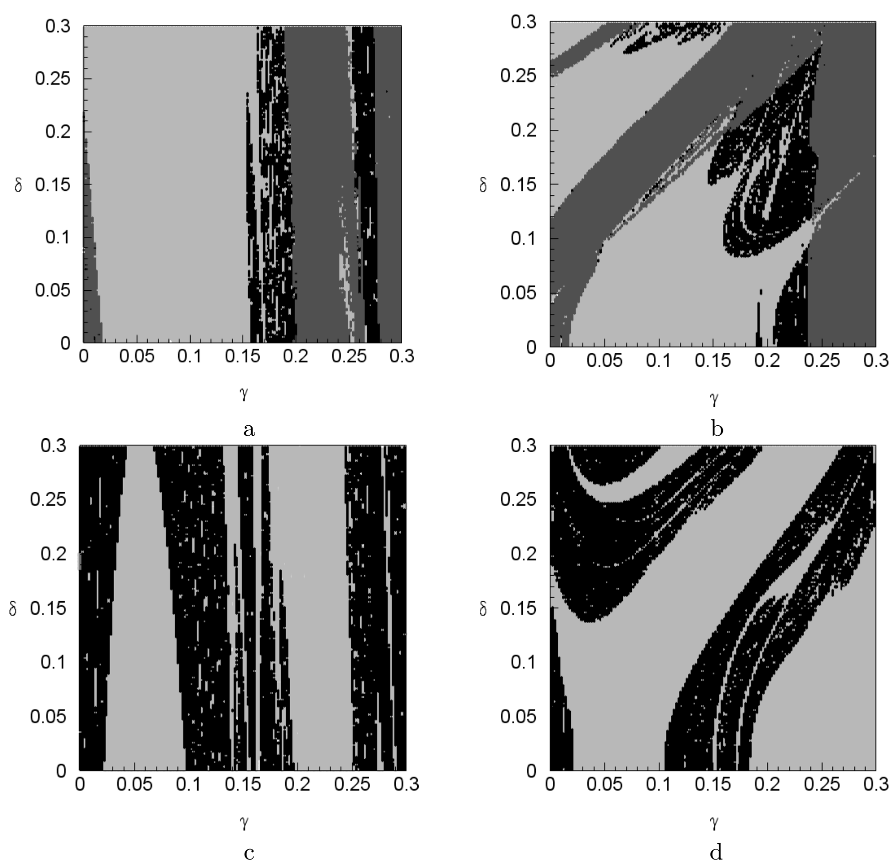


Fig. 1. Maps of dynamical regimes

cycles of the system. And finally, the black areas of the maps correspond to chaotic attractors.

At small values of the delays both systems have stable equilibrium positions (dark-grey areas in the figures). With an increase of the delay values the region of stable equilibrium positions is replaced by the region of periodic regimes and then by the region of chaotic regimes. With further increase of the delays the alternation of these three types of dynamical regimes takes place.

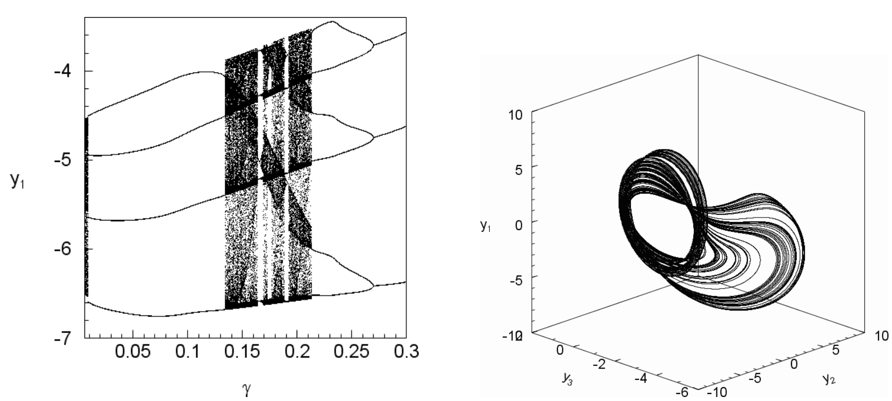
Let us study the dynamics of the systems (4) and (5) at other values of the parameters. The maps of dynamical regimes of three-dimensional system (4) and fifteen-dimensional system (5) at $C = -0.07$, $D = -0.5$, $E = -0.59$, $F = -0.15$ are built respectively in fig. 1 c, d. At small values of the delays both systems has chaotic attractors (black areas in the maps). With an increase of the delay of interaction between pendulum and electric motor γ the region of chaos is replaced by the region of periodic regimes. Then again chaos arises in the systems. Further this area is replaced by the area of limit cycles.

As seen from the constructed maps of dynamical regimes, the dynamics of three-dimensional system (4) and fifteen-dimensional system (5) is the same only at small values of the delays γ and δ . With an increase of the delays the differences of the dynamics of these systems is very significant.

The obtained maps of dynamical regimes allow us to conduct a quick qualitative identification of the type of steady-state regime of the systems (4) and (5). On the basis of constructed maps, more detailed studies of emerging dynamical regimes can be carried out. Particularly we can study the transition from regular to chaotic regimes [7], [8], [9].

4 Regular and chaotic dynamics

Let us study the influence of the delay values on chaotization of fifteen-dimensional system (4). We implement a horizontal section of the maps of dynamical regimes in fig.1d along the delay γ at $\delta = 0.1$. In other words, let us consider



a b

Fig. 2. Phase-parametric characteristic (a) and phase portrait of the chaotic attractor (b)

the behavior of the system (5) when parameters are $C = -0.07$, $D = -0.5$, $E = -0.59$, $F = -0.15$ and the delays $0 \leq \gamma \leq 0.3$ and $\delta = 0.1$. In fig. 2a phase-parametric characteristic of this system is shown. These figure illustrate the influence of the delay of interaction between pendulum and electric motor γ on steady-state dynamical regime change. At small values of γ the system (5) has chaotic attractors. Increasing the delay value the type of steady-state regime changes to periodic. The system's attractor is 4–turn limit cycle. At the further increase of the delay the transition to chaos happens under the scenario of Pomeau-Manneville, in a single bifurcation, through intermittency. The areas of chaos in the bifurcation tree (fig. 2a) are densely filled with points. Increasing the delay γ this area of chaos is replaced by periodic dynamical regimes. The system has 6-turn limit cycles and then 3-turn limit cycles. The transition to chaos in the system (5) happens either by Feigenbaum scenario (infinite cascade of period–doubling bifurcations of a limit cycle) or through intermittency.

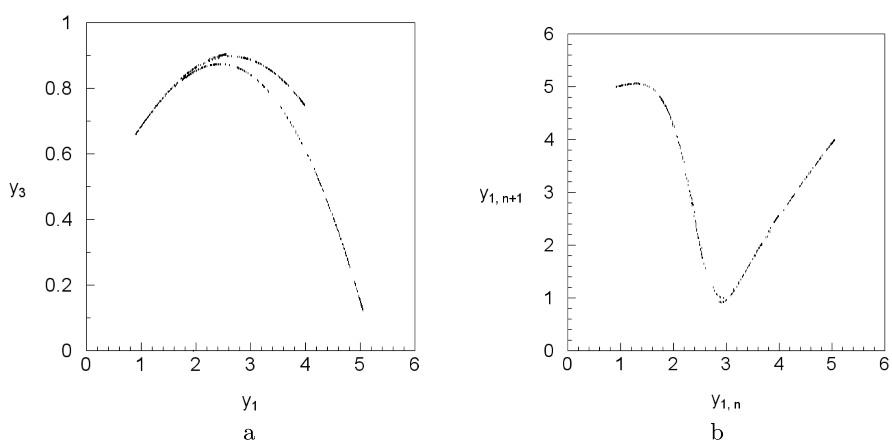


Fig. 3. Poincare section (a) and Poincare map (b) of the chaotic attractor

In fig. 2b phase portrait of typical chaotic attractor for the system (5) is depicted. This attractor was built at $\gamma = 0.15$. In fig. 3a, b respectively the Poincare section and the Poincare map of this chaotic attractor are shown. As can be seen from the fig. 3b the Poincare map have a structure that is close to a line on the plane and represent some chaotic set of points. Quantity of these points increases with increasing the time of numerical integration. It is impossible to foresee the order of points placement along the ribbons that form the map. However, it is known beforehand that they can only be placed along these ribbons. This allows building an analytical approximation of Poincare map in the form of discrete map. Such approximation of Poincare map of chaotic attractor of the system (1) was constructed in [7]. In the same way, the the fifteen-dimensional system (5) can be approximately reduced to a discrete map. The study of the dynamics of this map will be much easier than the dynamics of the initial system.

5 Conclusion

Taking into account various factors of delay in "pendulum-electric motor" systems is crucial. The presence of delay in such systems can affect qualitative change in the dynamic characteristics. In some cases the delay is the main reason of origination as well as vanishing of chaotic attractors. So delay is the controlling factor in the process of chaotization of pendulum systems.

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