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MODELING OF THE INFLUENCE OF DELAY FACTORS ON THE DYNAMICS OF NON-IDEAL PENDULUM SYSTEMS

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Abstract. The influence of various factors of delay on the dynamics of non-ideal dynamical systems "pendulum-electric motor" is considered. The approaches that allow to reduce the mathematical model of the system to the systems of differential equations without delay are proposed. The maps of dynamical regimes, the dependencies of maximal non-zero Lyapunov's characteristic exponent and phase-parametric characteristics of these systems are studied. The scenarios of transition from steady-state regular regimes to chaotic ones are identified.

Introduction

Mathematical models of pendulum systems are widely used to describe the dynamics of various oscillatory systems. Such models are used to study the oscillation of free liquid surface, membranes, various technical constructions, machines and mechanisms, in the study of cardiovascular system of live organisms, financial markets, etc.

Modern development of energy efficient and energy-preserving technologies requires the highest minimization of excitation source power of oscillatory systems. This leads to the fact that the energy of excitation source is comparable to the energy consumed by the oscillating system. Such systems as "source of excitation — oscillating subsystem" are called non-ideal by Zommerfeld-Kononenko [1]. In mathematical modeling of such systems, the limitation of excitation source power must be always taken into account.

Another important factor that significantly affects the change of steady-state regimes of dynamical systems, is the presence of different in their physical substance, factors of delay. The delay factors are always present in rather extended systems due to the limitations of signal transmission speed: stretching, waves of compression, bending, current strength, etc. In some cases, taking into account factors of delay leads only to minor quantitative changes in dynamic characteristics of pendulum systems. In other cases, taking into account these factors allow to identify qualitative changes in dynamic characteristics [2, 3].

The study of the influence of delay factors on the dynamical stability of equilibrium positions of pendulum systems was initiated by Yu. A. Mitropolsky in [4, 5]. But only ideal pendulum models were initially considered. In this paper non-ideal pendulum systems of the type "pendulum-electric motor" are considered. Mathematical models of such systems, in cases of absent delay factors, were obtained in [6]. In such systems the existence of chaotic attractors was discovered and proved that the main cause of chaos is limited excitation [6, 7].

Many aspects of constructing solutions of differential equations with delay and stability investigations of such equations are described in [8, 9, 10, 11, 12].

Mathematical model of the system

In [6] the equations of motion of the system "pendulum-electric motor" in the absence of any delay factors were obtained. They are

$$\begin{cases} \dfrac{dy_1}{d\tau} = Cy_1 - y_2y_3 - \dfrac{1}{8}(y_1^2y_2 + y_2^3); \\ \dfrac{dy_2}{d\tau} = Cy_2 + y_1y_3 + \dfrac{1}{8}(y_1^3 + y_1y_2^2) + 1; \\ \dfrac{dy_3}{d\tau} = Dy_2 + Ey_3 + F; \end{cases} \tag{1}$$

where phase variables y_1y_2 describe the deviation of the pendulum from the vertical and phase variable y_3 is proportional to the rotation speed of the motor shaft. The system parameters are defined by

$$C = -\delta_1\varepsilon^{-2/3}\omega_0^{-1}, D = -\frac{2ml^2}{I}, F = 2\varepsilon^{-2/3}(\frac{N_0}{\omega_0} + E) \tag{2}$$

where m — the pendulum mass, l — the reduced pendulum length, ω_0 — natural frequency of the pendulum, a — the length of the electric motor crank, $\varepsilon = \frac{a}{l}$, δ_1 — damping coefficient of the medium resistance force, I — the electric motor moment of inertia, E, N_0 — constants of the electric motor static characteristics.

Let us consider the following system of equations:

$$\begin{cases} \dfrac{dy_1(\tau)}{d\tau} = Cy_1(\tau-\delta) - y_2(\tau)y_3(\tau-\gamma) - \dfrac{1}{8}(y_1^2(\tau)y_2(\tau) + y_2^3(\tau)); \\ \dfrac{dy_2(\tau)}{d\tau} = Cy_2(\tau-\delta) + y_1(\tau)y_3(\tau-\gamma) + \dfrac{1}{8}(y_1^3(\tau) + y_1(\tau)y_2^2(\tau)) + 1; \\ \dfrac{dy_3(\tau)}{d\tau} = Dy_2(\tau-\gamma) + Ey_3(\tau) + F. \end{cases} \tag{3}$$

This system is a system of equations with constant delay. Positive constant parameter γ was introduced to account the delay effects of electric motor impulse on the pendulum. We assume that the delay of the electric motor response to the impact of the pendulum inertia force is also equal to γ. Taking into account the delay γ conditioned by the fact that the wave velocity perturbations on the elements of the construction has a finite value that depends on the properties of external fields, for instance, the temperature field. In turn, the constant positive parameter δ characterizes the delay of the medium reaction on the dynamical state of the pendulum. This delay is due to the limited sound velocity in that medium.

Let us consider two approaches that allow to reduce the time-delay system (3) to the systems of equations without delay. The first approach is as follows. Assuming a small delay, we can write

$$y_i(\tau-\gamma) = y_i(\tau) - \frac{y_1(\tau)}{d\tau}\gamma + ..., \quad i = 2, 3$$

$$y_i(\tau-\delta) = y_i(\tau) - \frac{y_1(\tau)}{d\tau}\delta + ..., \quad i = 1, 2$$

Then, if $C\delta \neq -1$, we get the following system of equations:

(4)
$$\begin{cases} \dot{y_1} = \dfrac{1}{1+C\delta}\left(Cy_1 - y_2\left[y_3 - \gamma\left(Dy_2 + Ey_3 + F\right)\right] - \dfrac{1}{8}(y_1^2 y_2 + y_2^3)\right); \\ \dot{y_2} = \dfrac{1}{1+C\delta}\left(Cy_2 + y_1 y_3 - y_1\gamma(Dy_2 + Ey_3 + F) + \dfrac{1}{8}(y_1^3 + y_1 y_2^2) + 1\right); \\ \dot{y_3} = (1 - C\gamma)Dy_2 - \dfrac{D\gamma}{8}(y_1^3 + y_1 y_2^2 + 8y_1 y_3 + 8) + Ey_3 + F. \end{cases}$$

The obtained system of equations is already a system of ordinary differential equations. Delays are included in this system as additional parameters.

In order to approximate the system (3) another, more precise, method can be used [8]. If $\gamma > 0,\ \delta > 0$ let us divide the segments $[-\gamma;\ 0]$ and $[-\delta;\ 0]$ into m equal parts. We introduce the following notation

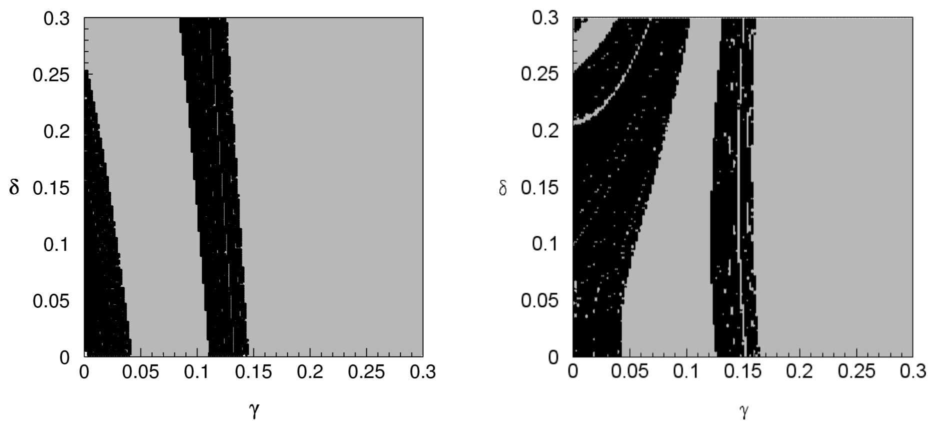
$$y_1(\tau - \tfrac{i\delta}{m}) = y_{1i}(\tau),\ y_2(\tau - \tfrac{i\gamma}{m}) = y_{2i}(\tau),\ y_2(\tau - \tfrac{i\delta}{m}) = \tilde{y}_{2i}(\tau),$$

$$y_3(\tau - \tfrac{i\gamma}{m}) = y_{3i}(\tau), i = \overline{0, m}.$$

Then, using difference approximation of derivative [8], [13] we obtain

$$(5) \quad \begin{cases} \dfrac{dy_{10}(\tau)}{d\tau} = Cy_{1m}(\tau) - y_{20}(\tau)y_{3m}(\tau) - \dfrac{1}{8}(y_{10}^2(\tau)y_{20}(\tau) + y_{20}^3(\tau)); \\ \dfrac{dy_{20}(\tau)}{d\tau} = C\tilde{y}_{2m}(\tau) + y_{10}(\tau)y_{3m}(\tau) + \dfrac{1}{8}(y_{10}^3(\tau) + y_{10}(\tau)y_{20}^2(\tau)) + 1; \\ \dfrac{dy_{30}(\tau)}{d\tau} = Dy_{2m}(\tau) + Ey_{30}(\tau) + F; \\ \dfrac{dy_{1i}(\tau)}{d\tau} = \dfrac{m}{\delta}(y_{1\ i-1}(\tau) - y_{1i}(\tau)), \quad i = \overline{1, m}; \\ \dfrac{dy_{2i}(\tau)}{d\tau} = \dfrac{m}{\gamma}(y_{2\ i-1}(\tau) - y_{2i}(\tau)), \quad i = \overline{1, m}; \\ \dfrac{d\tilde{y}_{2i}(\tau)}{d\tau} = \dfrac{m}{\delta}(\tilde{y}_{2\ i-1}(\tau) - \tilde{y}_{2i}(\tau)), \quad i = \overline{1, m}; \\ \dfrac{dy_{3i}(\tau)}{d\tau} = \dfrac{m}{\gamma}(y_{3\ i-1}(\tau) - y_{3i}(\tau)), \quad i = \overline{1, m}. \end{cases}$$

It is a system of ordinary differential equations of $(4m + 3)$-th order. In the absence of one of the delays (γ or δ), using the same reasoning, the system (3) can be reduced to the systems of $(2m + 3)$-th order. As in the system (4), the delays are included in these systems as additional parameters. Choosing a



a b

Рис. 1. Maps of dynamical regimes

sufficiently large m in the system (5), the system (3) will be very well approximated by the system (5) [8]. We note that the solutions y_1, y_2, y_3 of the system (3) are described by the functions y_{10}, y_{20}, y_{30} of the system (5).

Therefore, we obtained three-dimensional (4) and fifteen-dimensional (5) models each describing the system of equations with delay (3). These models are the systems of non-linear differential equations, so in general the study of steady-state regimes can be carried out only by using numerical methods and algorithms. The methodology of such studies is described in detail in [6].

Maps of dynamical regimes

A very clear picture of the dynamical system behavior can give us a map of dynamical regimes. It is a diagram on the plane, where two parameters are plotted on axes and the boundaries of different dynamical regimes areas are shown. The construction of dynamical regimes maps is based on analysis and processing of spectrum of Lyapunov characteristic exponents [6, 14]. Where necessary, for more accurate determination of steady-state regime of the system, we study other characteristics of attractors: Poincare sections and maps, Fourier spectrums, phase portraits and distributions of the invariant measure.

Let us consider the behavior of the systems (4) and (5) when the parameters are $C = -0.1$, $D = -0.53$, $E = -0.6$, $F = 0.19$. In fig. 1 the maps of dynamical regimes are shown. The map in fig. 1a was built for three-dimensional model (4) and the map in fig. 1b was built for fifteen-dimensional model (5). These figures illustrate the effect of delays γ and δ on changing the type of steady-state regime of the systems. The dark-grey areas of the maps correspond to equilibrium positions of the system. The light-grey areas of the maps correspond to limit cycles of the system. And finally, the black areas of the maps correspond to chaotic attractors.

We can notice a certain similarity the maps in fig.1a, b. At small values of the delays both systems have chaotic attractors (black areas in the figures). With

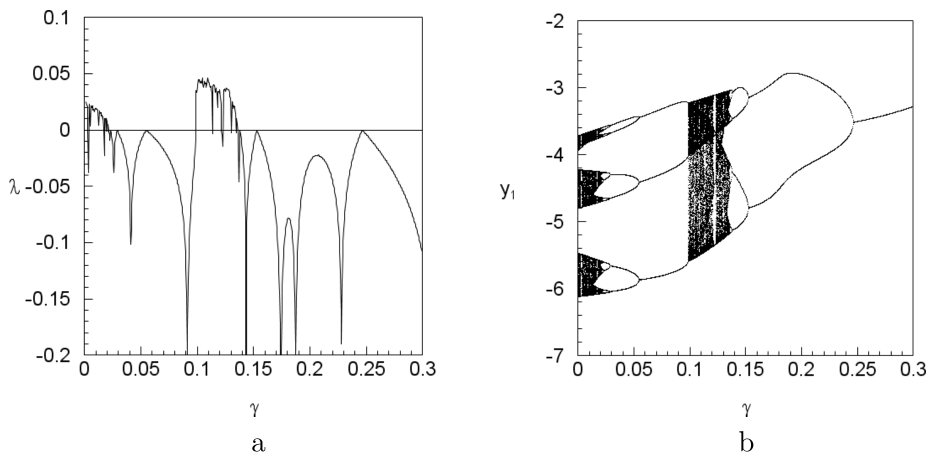


Рис. 2. The dependence of maximal non-zero Lyapunov's characteristic exponent (a), phase-parametric characteristic (b) of three-dimensional system (4)

an increase of the delay values the region of chaos is replaced by the region of periodic regimes (light-grey areas in the figures). Then again chaos arises in the system. Further this area is replaced by the area of limit cycles.

As seen from the constructed maps of dynamical regimes, the dynamics of the system (4) and (5) is the same only at small values of the delay γ and δ. Increasing the delays the differences of the dynamics of these systems are very significant.

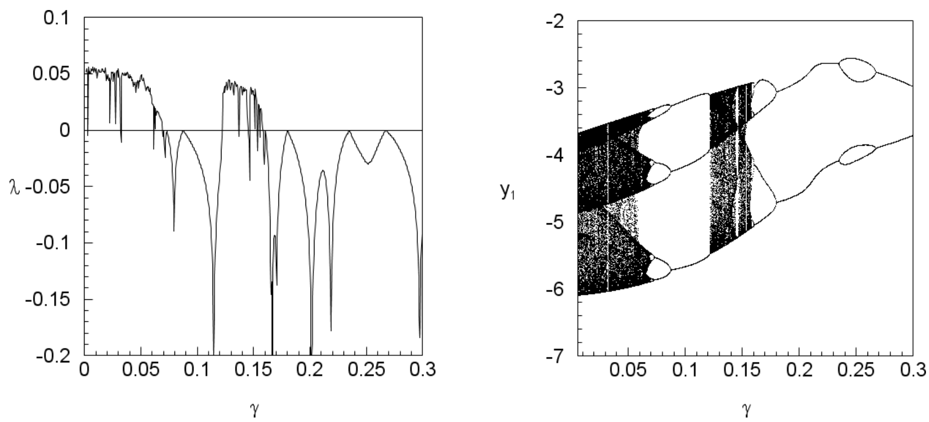
Regular and chaotic dynamics

Let us study the types of regular and chaotic attractors that exist in the systems (4) and (5). We implement a horizontal section of the maps of dynamical regimes in fig.1a, b along the delay γ at $\delta = 0.15$. In other words, let us consider the behavior of the systems (4) and (5) when parameters are $C = -0.1$, $D = -0.53$, $E = -0.6$, $F = 0.19$ and the delays $\delta = 0.15$ and $0 \leq \gamma \leq 0.3$.

In fig. 2a,b the dependence of maximum non-zero Lyapunov's characteristic exponent and phase-parametric characteristic of three-dimensional system (4) are shown respectively. These figures illustrate the influence of the delay of interaction between pendulum and electric motor γ on chaotization of the system (4).

Let us construct the same characteristics at the same values of the parameters for fifteen-dimensional system (5). In fig. 3a,b respectively the dependence of maximum non-zero Lyapunov's characteristic exponent and phase-parametric characteristic are shown.

In fig.2a, 3a we can clearly see the presence of intervals γ in which maximum Lyapunov exponent of the systems is positive. In these intervals the systems have chaotic attractors. The area of existence of chaos is clearly seen in phase-parametric characteristics of the systems. The areas of chaos in the bifurcation trees are densely filled with points. A careful examination of the obtained



a b

Рис. 3. The dependence of maximal non-zero Lyapunov's characteristic exponent (a), phase-parametric characteristic (b) of fifteen-dimensional system (5)

images allows not only to identify the origin of chaos in the systems, but also to describe the scenario of transition to chaos. So with a decrease of γ there are the transitions to chaos by Feigenbaum scenario (infinite cascade of period-doubling bifurcations of a limit cycle). Bifurcation points for the delay γ are clearly visible in each figures. These points are the points of approaches of the Lyapunov's exponent graph to the zero line (fig.2a, 3a) and the points of splitting the branches of the bifurcation tree (fig.2b, 3b). In turn, the transition to chaos with an increase of the delay happens under the scenario of Pomeau-Manneville, in a single bifurcation, through intermittency.

A careful analysis of these figures allows to see qualitative similarity of the respective characteristics of the systems (4) and (5). However, with increasing the delay the differences in the dynamics of these systems become very significant. So for instance at $\gamma = 0.05$ the steady–state regime of the system (4) is limit cycle. While at this value of the delay the attractor of the system (5) is chaotic attractor. Conversely, for example at $\gamma = 0.11$ the system (4) has steady–state chaotic regime. While at this value of the delay the system (5) has periodic regime of oscillations.

This suggests that three-dimensional system of equations (4) should be used to study the system (3) only at very small values of the delay. With increasing values of the delay to study regular and chaotic oscillations of "pendulum-electric motor" system, fifteen-dimensional system of equations (5) should be used.

Conclusion

Taking into account various factors of delay in "pendulum-electric motor" systems is crucial. The presence of delay in such systems can affect the qualitative change in the dynamic characteristics. It is shown, that in some cases the delay is the main reason of origination as well as vanishing of chaotic attractors.

It is shown that for small values of the delay it is sufficient to use three-dimensional mathematical models, whereas for relatively high values of the delay the fifteen-dimensional mathematical model should be used.

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